

## ISOPERIMETRIC INEQUALITY UNDER KÄHLER RICCI FLOW

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ABSTRACT. Let  $(\mathbf{M}, g(t))$  be a Kähler Ricci flow with positive first Chern class. First, we prove a uniform isoperimetric inequality for all time. Second it is shown that the Ricci potential is in the class  $C^{1,\alpha}(B)$  when the ball  $B$  is within a coordinate chart. Here  $\alpha$  is any positive number less than 1. We also prove a Cheng-Yau type log gradient bound for positive harmonic functions on  $(\mathbf{M}, g(t))$  without assuming the Ricci curvature is bounded from below.

## 1. INTRODUCTION

In this paper we study Kähler Ricci flows

$$(1.1) \quad \partial_t g_{i\bar{j}} = -R_{i\bar{j}} + g_{i\bar{j}} = \partial_i \partial_{\bar{j}} u, \quad t > 0,$$

on a compact, Kähler manifold  $\mathbf{M}$  of complex dimension  $m = n/2$ , with positive first Chern class.

Given initial Kähler metric  $g_{i\bar{j}}(0)$ , H. D. Cao [Ca] proved that (1.1) has a solution for all time  $t$ . Recently, many results concerning long time and uniform behavior of (1.1) have appeared. For example, when the curvature operator or the bisectional curvature is nonnegative, it is known that solutions to (1.1) stays smooth when time goes to infinity (see [CCZ], [CT1] and [CT2] for examples). In the general case, Perelman (cf [ST]) proved that the scalar curvature  $R$  is uniformly bounded, and the Ricci potential  $u(\cdot, t)$  is uniformly bounded in  $C^1$  norm, with respect to  $g(t)$ . When the complex dimension  $m = 2$ , let  $(\mathbf{M}, g(t))$  be a solution to (1.1), it is proved in ([CW]) that the isoperimetric constant for  $(\mathbf{M}, g(t))$  is bounded from below by a uniform constant. We mention that an isoperimetric estimate for the Ricci flow on the two sphere was already proven by Hamilton in [Ha].

In this paper, we prove that in all complex dimensions, the isoperimetric constant for  $(\mathbf{M}, g(t))$  is bounded from below by a uniform constant. This extends the result of Chen-Wang mentioned above. We also prove that for any  $\alpha \in (0, 1)$ , the Ricci potential  $u(\cdot, t)$  is in the class  $C^{1,\alpha}(B, g(t))$  when the ball  $B$  is within a coordinate chart. These two results seem to add more weight to the belief that the Kähler Ricci flow converges to a Kähler Ricci soliton as  $t \rightarrow \infty$ , except on a subvariety of complex codimension 2,

To make the statement precise, let's introduce notations and definition. We use  $\mathbf{M}$  to denote a compact Riemann manifold and  $g(t)$  to denote the metric at time  $t$ ;  $d(x, y, t)$  is the geodesic distance under  $g(t)$ ;  $B(x, r, t) = \{y \in \mathbf{M} \mid d(x, y, t) < r\}$  is the geodesic ball of radius  $r$ , under metric  $g(t)$ , centered at  $x$ , and  $|B(x, r, t)|_{g(t)}$  is the volume of  $B(x, r, t)$  under  $g(t)$ ;  $dg(t)$  is the volume element. We also reserve  $R = R(x, t)$  as the scalar curvature under  $g(t)$ . When the time variable  $t$  is not explicitly used, we may also suppress it in the notations mentioned above.

The main results of the paper are the following theorems.

**Theorem 1.1.** *Let  $(\mathbf{M}, g(t))$ ,  $\partial_t g_{i\bar{j}} = -R_{i\bar{j}} + g_{i\bar{j}}$ , be a Kähler Ricci flow on a  $n$  real dimensional compact, Kähler manifold with positive first Chern class. Then there exists a uniform constant  $S_0$ , depending only on the initial metric  $g(0)$  and a numerical constant  $C$ , such that*

$$\left[ \int_{\mathbf{M}} |u|^{n/(n-1)} dg(t) \right]^{(n-1)/n} \leq S_0 \int_{\mathbf{M}} |\nabla u| dg(t) + \frac{C}{|\mathbf{M}|_{g(t)}^{1/n}} \int_{\mathbf{M}} |u| dg(t)$$

for all  $u \in C^\infty(\mathbf{M})$ .

**Theorem 1.2.** *Let  $(\mathbf{M}, g(t))$ ,  $\partial_t g_{i\bar{j}} = -R_{i\bar{j}} + g_{i\bar{j}}$ , be a Kähler Ricci flow on a compact, Kähler manifold with positive first Chern class. Let  $u$  be the Ricci potential, i.e.  $u$  satisfies, in a local coordinates,*

$$\partial_i \partial_{\bar{j}} u = g_{i\bar{j}} - R_{i\bar{j}}.$$

*Suppose the ball  $B(x_0, r_0, t)$  is contained in a coordinate chart. Then for any fixed  $\alpha \in (0, 1)$ , there exists constant  $C_0$ , depending only on the initial metric  $g(0)$ ,  $r_0$  and  $\alpha$ , such that*

$$\|\nabla u\|_{C^\alpha(B(x_0, r_0/2, t))} \leq C_0.$$

Here the gradient  $\nabla$  and the  $C^\alpha$  norm are with respect to the metric  $g(t)$ .

*Remark.*

1. It is well known that Theorem 1.1 implies a uniform lower bound for the isoperimetric constant of  $(\mathbf{M}, g(t))$ , i.e. there exists a positive constant  $c_0$ , depending only on the initial metric such that

$$I(\mathbf{M}, g(t)) \equiv \inf_{D \subset \mathbf{M}} \frac{|\partial D|}{[\min\{|D|, |\mathbf{M} - D|\}]^{(n-1)/n}} \geq c_0.$$

Here all the volume are with respect to  $g(t)$ ; and  $D$  is a subdomain of  $\mathbf{M}$  such that  $\partial D$  is a  $n-1$  dimensional submanifold of  $\mathbf{M}$ . A proof can be found in [CLN] Section 5.1 e.g.

2. It remains to be seen whether one can get the  $C^{1,\alpha}$  bounds on the Ricci potential  $u$  without the condition that the ball  $B(x_0, r_0, t)$  is contained in a coordinate chart. It is known that volume noncollapsing and curvature tensor bound implies such a condition.

The proof of the theorems are based on the following properties for Kähler Ricci flow on a compact manifold with positive first Chern class.

*Property A.* Let  $(\mathbf{M}, g(t))$  be a Kähler Ricci flow (1.1) on a compact manifold with positive first Chern class. There exist uniform positive constants  $C$ , and  $\kappa$  so that

1.  $|R(g(t))| \leq C$ ,
2.  $\text{diam}(\mathbf{M}, g(t)) \leq C$ ,
3.  $\|u\|_{C^1} \leq C$ .
4.  $|B(x, r, t)|_{g(t)} \geq \kappa r^n$ , for all  $t > 0$  and  $r \in (0, \text{diam}(\mathbf{M}, g(t)))$ .
5.  $|B(x, r, t)|_{g(t)} \leq \kappa^{-1} r^n$  for all  $r > 0$ ,  $t > 0$ .

*Property B.* Under the same assumption as in Property A, there exists a uniform constant  $S_2$  so that the following  $L^2$  Sobolev inequality holds:

$$\left( \int_{\mathbf{M}} v^{2n/(n-2)} dg(t) \right)^{(n-2)/n} \leq S_2 \left( \int_{\mathbf{M}} |\nabla v|^2 dg(t) + \int_{\mathbf{M}} v^2 dg(t) \right)$$

for all  $v \in C^\infty(\mathbf{M}, g(t))$ .

Property A 1-4 is due to Perelman (c.f. [ST]), Property B was first established in [Z07] (see also [Ye], [Z10]). Property A 5 can be found in [Z11] and also [CW2].

The rest of the paper is organized as follows. In Section 2, we prove some gradient bounds for harmonic functions on  $(\mathbf{M}, g(t))$ . Since the bounds do not rely on the usual lower bound of Ricci curvature, the result may be of independent interest. Using these bounds, we prove the two theorems in Section 3.

## 2. GRADIENT BOUNDS FOR HARMONIC FUNCTIONS

In order to prove the theorems, in this section we state and prove a number of results on harmonic functions on certain manifolds with fixed metric. These results are well known when the manifold has nonnegative Ricci curvature, a property that is not available for us. Since some of these results may be of independent interest, we will also deal with the real variable case and impose some conditions which are more general than needed for the proof of the theorems in Section 1. As the metric is independent of time in this section, we will suppress the time variable  $t$ .

In this section, the basic assumptions on the  $n$  real dimensional manifolds  $\mathbf{M}$  are

*Assumption 1.  $L^2$  Sobolev inequality: there is a positive constant  $\alpha$  such that*

$$\left( \int_{\mathbf{M}} u^{2n/(n-2)} dg(t) \right)^{(n-2)/n} \leq \alpha \left( \int_{\mathbf{M}} |\nabla u|^2 dg(t) + \int_{\mathbf{M}} u^2 dg(t) \right)$$

for all  $u \in C^\infty(\mathbf{M})$ .

*Assumption 2. There exists a positive constant  $\kappa$ , such that*

$$\kappa r^n \leq |B(x, r)| \leq \kappa^{-1} r^n, \quad x \in \mathbf{M}, \quad 0 < r < \text{diam}(\mathbf{M}) \leq 1.$$

*Assumption 3. There exists a smooth function  $L = L(x)$  and 2 smooth parallel  $(2, 2)$  tensor fields  $P$  and  $Q$  such that the Ricci curvature is given by*

$$R_{ij} = P_{ij}^{kl} \partial_k \partial_l L + Q_{ij}^{kl} g_{kl}$$

under a local coordinates. Moreover  $\|P\|_\infty \leq 1$ ,  $\|Q\|_\infty \leq 1$ . Here  $\partial_k \partial_l L$  is the Hessian of  $L$ .

Note that Assumption 3 includes as a special case, the formula for the Ricci curvature on Kähler manifolds  $\partial_i \partial_{\bar{j}} u = g_{i\bar{j}} - R_{i\bar{j}}$  where  $u$  is the Ricci potential.

**Lemma 2.1.** *Suppose  $(\mathbf{M}, g)$  is a compact Riemann manifold of real dimension  $n$ , satisfying Assumptions 1, 2, 3.*

*Let  $u$  be a smooth harmonic function in  $B(x_0, r)$  where  $x_0 \in \mathbf{M}$  and  $r \leq \text{diam}(\mathbf{M})$ . Then there exists a positive constant  $C_0 = C_0(\alpha, \kappa, \|L\|_\infty)$  such that*

$$\sup_{x \in B(x_0, r/2)} |\nabla u(x)| \leq C_0 \frac{1}{r} \left( \frac{1}{r^n} \int_{B(x_0, r)} u^2 dg \right)^{1/2}.$$

*Proof.* Since  $u$  solves  $\Delta u = 0$ , by Bochner's formula, we have

$$(2.1) \quad \Delta |\nabla u|^2 = 2|Hess u|^2 + 2Ric(\nabla u, \nabla u).$$

Given  $\sigma \in (0, 1)$ , let  $\psi = \psi(x)$  be a standard smooth cut-off function such that  $\psi(x) = 0$  when  $x \in B(x_0, r)^c$ ;  $0 \leq \psi \leq 1$  and  $\psi(x) = 1$  when  $x \in B(x_0, \sigma r)$  and  $|\nabla \psi| \leq \frac{4}{(1-\sigma)r}$ .

For clarity of presentation, we write

$$f = |\nabla u|^2.$$

Using  $f\psi^2$  as a test function on (2.1), after a routine calculation, we derive

$$(2.2) \quad \begin{aligned} & \int_{B(x_0, r)} |\nabla(f\psi)|^2 dg \\ & \leq \frac{C}{(1-\sigma)^2 r^2} \int_{B(x_0, r)} f^2 dg - 2 \int_{B(x_0, r)} |Hess u|^2 |\nabla u|^2 \psi^2 dg - 2 \int_{B(x_0, r)} Ric(\nabla u, \nabla u) |\nabla u|^2 \psi^2 dg \\ & \equiv I_1 + I_2 + I_3. \end{aligned}$$

Now we want to absorb part of  $I_3$  by  $I_2$  which is a good term. In a local orthonormal coordinates, we denote  $u_i$  the  $i$ -th component of  $\nabla u$ . Then By Assumption 3, we have, after integrating by parts,

$$\begin{aligned} I_3 &= -2 \int R_{ij} u_i u_j f \psi^2 dg \\ &= -2 \int P_{ij}^{kl} (\partial_k \partial_l L) u_i u_j f \psi^2 dg - 2 \int Q_{ij}^{kl} g_{kl} u_i u_j f \psi^2 dg \\ &= 2 \int P_{ij}^{kl} (\partial_l L) (\partial_k u_i) u_j f \psi^2 dg + 2 \int P_{ij}^{kl} (\partial_l L) u_i (\partial_k u_j) f \psi^2 dg + 2 \int P_{ij}^{kl} (\partial_l L) u_i u_j \partial_k (f \psi^2) dg \\ &\quad - 2 \int Q_{ij}^{kl} g_{kl} u_i u_j f \psi^2 dg. \end{aligned}$$

Here we also used the assumption that the  $P$  tensor is parallel. From here, using Young's inequality, it is easy to see that

$$I_3 \leq \frac{1}{2} \int |\nabla(f\psi)|^2 dg + \int |Hess u|^2 |\nabla u|^2 \psi^2 dg + C \left( \frac{\|L\|_\infty^2}{[(1-\sigma)r]^2} + \|L\|_\infty^2 + 1 \right) \int f^2 \psi^2 dg.$$

Substituting this to (2.2), we deduce

$$\int |\nabla(f\psi)|^2 dg \leq C \left( \frac{\|L\|_\infty^2}{[(1-\sigma)r]^2} + \|L\|_\infty^2 + 1 \right) \int f^2 \psi^2 dg.$$

Since  $diam(\mathbf{M}) \leq 1$  by Assumption 2, the last inequality implies

$$\int |\nabla(f\psi)|^2 dg \leq C \frac{\|L\|_\infty^2 + 1}{[(1-\sigma)r]^2} \int f^2 \psi^2 dg.$$

Using the  $L^2$  Sobolev inequality in Assumption 1 and Moser's iteration, we deduce that

$$(2.3) \quad \sup_{x \in B(x_0, r/2)} |\nabla u(x)|^2 \leq C_1 \frac{1}{r^n} \int_{B(x_0, 3r/4)} |\nabla u|^2 dg$$

where  $C_1 = C_1(\alpha, \kappa, \|L\|_\infty)$ .

Next we take  $\sigma = 3/4$  in the definition of the cut off function  $\psi$ . Using  $u\psi^2$  as a test function on  $\Delta u = 0$ , we infer, after a routine calculation

$$\int_{B(x_0, 3r/4)} |\nabla u|^2 dg \leq \frac{C}{r^2} \int_{B(x_0, r)} u^2 dg.$$

Here  $C$  is a numerical constant. Combining the last two inequalities we arrive at

$$\sup_{x \in B(x_0, r/2)} |\nabla u(x)|^2 \leq C_0 \frac{1}{r^{n+2}} \int_{B(x_0, r)} u^2 dg$$

where  $C_0 = C_0(\alpha, \kappa, \|L\|_\infty)$ .  $\square$

The next lemma is simply the  $L^2$  mean value inequality for the Laplace and heat equation under Assumptions 1 and 2. Since the result is well known (Grigoryan [Gr] and Saloff-Coste [Sa]), we omit the proof.

**Lemma 2.2.** *Let  $\mathbf{M}$  be a manifold satisfying Assumptions 1, 2.*

*Suppose  $u$  be is a smooth harmonic function in  $B(x_0, r)$  where  $x_0 \in \mathbf{M}$  and  $r \leq \text{diam}(\mathbf{M})$ . Then there exists a positive constant  $C_1 = C_1(\alpha, \kappa)$  such that*

$$\sup_{x \in B(x_0, r/2)} |u(x)| \leq C_1 \left( \frac{1}{r^n} \int_{B(x_0, r)} u^2 dg \right)^{1/2}.$$

*Suppose  $u$  is a solution of the heat equation  $\Delta u - \partial_t u = 0$  in the space time cube  $B(x_0, r) \times [t_0 - r^2, t_0]$ . Then*

$$\sup_{(x, t) \in B(x_0, r/2) \times [t_0 - r^2/4, t_0]} |u(x, t)| \leq C_1 \left( \frac{1}{r^{n+2}} \int_{t_0 - r^2}^{t_0} \int_{B(x_0, r)} u^2 dg ds \right)^{1/2}.$$

The next lemma provides bounds for the Green's function of the Laplacian and its gradients.

**Lemma 2.3.** *Let  $\mathbf{M}$  be a manifold satisfying Assumptions 1, 2 and 3. Assume also  $\text{diam}(\mathbf{M}) > \beta > 0$  for a positive constant  $\beta$ . Let  $\Gamma_0$  be the Green's function of the Laplacian  $\Delta$  on  $\mathbf{M}$ . Then there exists a positive constant  $C_0 = C_0(\alpha, \beta, \kappa, \|L\|_\infty)$  such that*

- (a).  $|\Gamma_0(x, y)| \leq \frac{C_0}{d(x, y)^{n-2}}, x, y \in \mathbf{M},$
- (b).  $|\nabla_x \Gamma_0(x, y)| \leq \frac{C_0}{d(x, y)^{n-1}}, x, y \in \mathbf{M}.$

*Proof.* Once (a) is proven, (b) is just a consequence of (a) and Lemma 2.1 applied on the ball  $B(x, d(x, y)/2)$ . So now we just need to prove (a).

On a compact manifold  $\mathbf{M}$ , we know that

$$(2.4) \quad \Gamma_0(x, y) = \int_0^\infty \left( G(x, t, y) - \frac{1}{|\mathbf{M}|} \right) dt$$

where  $G$  is the fundamental solution of the heat equation  $\Delta u - \partial_t u = 0$ . We remark that the metric is fixed here. So we need to bound  $G$ . Under Assumptions 1 and 2, Grigoryan [Gr] and Saloff-Coste [Sa] proved that there exist positive constants  $A_1, A_2, A_3$  which depend only on  $\alpha$  and  $\kappa$  such that

$$(2.5) \quad G(x, t, y) \leq A_1 \left( 1 + \frac{1}{t^{n/2}} \right) e^{-A_2 d(x, y)^2/t}.$$

Moreover, by [Sa], the following  $L^2$  Poincaré inequality holds: for any  $u \in C^\infty(\mathbf{M})$ ,  $r \in (0, \text{diam}(\mathbf{M}))$ ,

$$(2.6) \quad \int_{B(x, r/2)} |u - \bar{u}_{B(x, r/2)}|^2 dg \leq A_3 r^2 \int_{B(x, r)} |\nabla u|^2 dg.$$

By a trick in Jerison [J], which uses only volume doubling property, one has that

$$\int_{B(x, r)} |u - \bar{u}_{B(x, r)}|^2 dg \leq C A_3 r^2 \int_{B(x, r)} |\nabla u|^2 dg.$$

Here  $C$  depends only on  $\kappa$ .

Let  $u_0 \in C^\infty(\mathbf{M})$  be a function such that  $\int_{\mathbf{M}} u_0 dg = 0$ . Then the function

$$(2.7) \quad u(x, t) = \int_{\mathbf{M}} \left( G(x, t, z) - \frac{1}{|\mathbf{M}|} \right) u_0(z) dg(z)$$

is a solution to the heat equation such that  $\int_{\mathbf{M}} u(x, t) dg(x) = 0$ . By the  $L^2$  Poincaré inequality with  $r = \text{diam}(\mathbf{M})$ , we have

$$\int_{\mathbf{M}} u^2 dg \leq C A_3 \text{diam}(\mathbf{M})^2 \int_{\mathbf{M}} |\nabla u|^2 dg \leq C A_3 \int_{\mathbf{M}} |\nabla u|^2 dg$$

since  $\text{diam}(\mathbf{M}) \leq 1$  by assumption. From this we deduce

$$\frac{d}{dt} \int_{\mathbf{M}} u^2 dg = -2 \int_{\mathbf{M}} |\nabla u|^2 dg = -2(C A_3)^{-1} \int_{\mathbf{M}} u^2 dg$$

and consequently

$$\int_{\mathbf{M}} u^2(z, s) dg \leq e^{-2(C A_3)^{-1}s} \int_{\mathbf{M}} u_0^2(z) dg, \quad s > 0.$$

Recall that we assume  $\text{diam}(\mathbf{M}) > \beta > 0$ . For  $t \geq \beta^2$ , we can apply Lemma 2.2 to get

$$u^2(x, t) \leq C_1^2 \frac{1}{\beta^{n+2}} \int_{t-\beta^2}^t \int_{\mathbf{M}} u^2(z, s) dg ds.$$

Combining this with the previous inequality, we arrive at

$$u^2(x, t) \leq C_2 e^{-2(C A_3)^{-1}t} \int_{\mathbf{M}} u_0^2(z) dg$$

where  $C_2 = C_0(\alpha, \beta, \kappa, A_3)$ . By (2.7), this means

$$\left[ \int_{\mathbf{M}} \left( G(x, t, z) - \frac{1}{|\mathbf{M}|} \right) u_0(z) dg \right]^2 \leq C_2 e^{-2(C A_3)^{-1}t} \int_{\mathbf{M}} u_0^2(z) dg$$

Fixing  $x \in \mathbf{M}$  and  $t \geq \beta^2$ , and taking  $u_0(z) = G(x, t, z) - \frac{1}{|\mathbf{M}|}$  in the above inequality, we obtain

$$(2.8) \quad \int_{\mathbf{M}} \left( G(x, t, z) - \frac{1}{|\mathbf{M}|} \right)^2 dg \leq C_2 e^{-2(C A_3)^{-1}t}, \quad t \geq \beta^2.$$

Fixing  $x$ , the function  $h(z, t) \equiv G(x, t, z) - \frac{1}{|\mathbf{M}|}$  is also a solution to the heat equation. Applying the mean value inequality in Lemma 2.2 on the cube  $B(y, \beta) \times [t - \beta^2, t]$ , we infer

$$h^2(y, t) \leq C_1^2 \frac{1}{\beta^{n+2}} \int_{t-\beta^2}^t \int_{\mathbf{M}} h^2(z, s) dg ds.$$

That is

$$\left( G(x, t, y) - \frac{1}{|\mathbf{M}|} \right)^2 \leq C_1^2 \frac{1}{\beta^{n+2}} \int_{t-\beta^2}^t \int_{\mathbf{M}} \left( G(x, s, z) - \frac{1}{|\mathbf{M}|} \right)^2 dg ds.$$

Substituting (2.8) to the last inequality, we deduce

$$(2.9) \quad \left| G(x, t, y) - \frac{1}{|\mathbf{M}|} \right| \leq C_3 e^{-C_4 t}, \quad t \geq \beta^2,$$

where  $C_3, C_4$  depend only on  $\alpha, \beta, \kappa$  and  $A_3$  which only depends on  $\alpha, \kappa$ .

From (2.4),

$$\begin{aligned} \Gamma_0(x, y) &= \int_0^{\beta^2} \left( G(x, t, y) - \frac{1}{|\mathbf{M}|} \right) dt + \int_{\beta^2}^{\infty} \left( G(x, t, y) - \frac{1}{|\mathbf{M}|} \right) dt \\ &\equiv I_1 + I_2. \end{aligned}$$

Using the bound (2.5) on  $I_1$  and (2.9) on  $I_2$ , we derive, after simple integration,

$$|\Gamma_0(x, y)| \leq \frac{C_0}{d(x, y)^{n-2}},$$

where  $C_0$  depends only on  $\alpha, \beta, \kappa$ . This proves part (a) of the Lemma.

As mentioned earlier, part (b) follows from part (a) and Lemma 2.1.  $\square$

The next result is a Cheng-Yau type log gradient estimate. Although not used in the proof of the theorems, it may be of independent interest.

**Proposition 2.1.** *Let  $\mathbf{M}$  be a manifold satisfying Assumptions 1, 2 and 3. Let  $u$  be a positive harmonic function in the geodesic ball  $B(x, 2r)$ , which is properly contained in  $\mathbf{M}$ . Then there exists a positive constant  $C$ , depending only on the controlling constants in Assumptions 1-3, such that*

$$\sup_{B(x, r)} |\nabla \ln u| \leq \frac{C}{r}$$

when  $r \in (0, 1]$ .

*Proof.*

For convenience, we use the following notations

$$h \equiv \ln u, \quad F \equiv |\nabla h|^2.$$

Following [CY], it is well known that  $\Delta h = -F$  and

$$\Delta F = -2\nabla h \nabla F + 2|Hess h|^2 + 2Ric(\nabla h, \nabla h).$$

Consider the function

$$(2.10) \quad w \equiv F^{5n}.$$

By a routine calculation, we know that, for any  $p \geq 1$ ,

$$(2.11) \quad \Delta w^p \geq -2\nabla h \nabla w^p + 10np F^{5np-1} |Hess h|^2 + 10np F^{5np-1} Ric(\nabla h, \nabla h)$$

Given  $\sigma \in (0, 1)$ , let  $\psi = \psi(x)$  be a standard smooth cut-off function such that  $\psi(x) = 0$  when  $x \in B(x_0, r)^c$ ;  $0 \leq \psi \leq 1$  and  $\psi(x) = 1$  when  $x \in B(x_0, \sigma r)$  and  $|\nabla \psi| \leq \frac{4}{(1-\sigma)r}$ . Using  $w^2\psi$  as a test function on (2.11), we deduce, after a straight forward calculation, that

$$(2.12) \quad \begin{aligned} \int |\nabla(w^p\psi)|^2 dg &\leq -10np \int F^{5np-1} |Hess h|^2 w^p \psi^2 dg + 2 \int \nabla h \nabla w^p w^p \psi^2 dg \\ &\quad - 10np \int F^{5np-1} Ric(\nabla h, \nabla h) w^p \psi^2 dg \\ &\equiv I_1 + I_2 + I_3. \end{aligned}$$

Next we will show that the negative term  $I_1$  dominate  $I_2$  and  $I_3$ , modulo some harmless terms. Observe that

$$\begin{aligned} I_2 &= \int \psi^2 \nabla h \nabla w^{2p} dg \\ &= -2 \int \psi \nabla \psi \nabla h w^{2p} dg - \int \psi^2 \Delta h w^{2p} dg. \end{aligned}$$

Recall that  $\Delta h = -|\nabla h|^2 = F$ . Hence, by Young's inequality, for any given  $\epsilon > 0$ ,

$$(2.13) \quad I_2 \leq (\epsilon + 1) \int F w^{2p} \psi^2 dg + \epsilon^{-1} \|\nabla \psi\|_\infty^2 \int w^{2p} \psi^2 dg.$$

It takes a little longer to prove the bound for  $I_3$ . By our condition on the Ricci curvature  $R_{ij}$ , we have

$$I_3 = -10np \int F^{5np-1} (P_{ij}^{kl} \partial_k \partial_l L + Q_{ij}^{kl} g_{kl}) \partial_i h \partial_j h w^p \psi^2 dg.$$

After integration by parts, this becomes

$$(2.14) \quad \begin{aligned} I_3 &= 10np(5np-1) \int F^{5np-2} \partial_k F P_{ij}^{kl} \partial_l L \partial_i h \partial_j h w^p \psi^2 dg \\ &\quad + 10np \int F^{5np-1} P_{ij}^{kl} \partial_l L (\partial_k \partial_i h) \partial_j h w^p \psi^2 dg \\ &\quad + 10np \int F^{5np-1} P_{ij}^{kl} \partial_l L \partial_i h (\partial_k \partial_j h) w^p \psi^2 dg \\ &\quad + 10np \int F^{5np-1} P_{ij}^{kl} \partial_l L \partial_i h \partial_j h \partial_k (w^p \psi) \psi dg \\ &\quad + 10np \int F^{5np-1} P_{ij}^{kl} \partial_l L \partial_i h \partial_j h w^p \psi \partial_k \psi \\ &\quad - 10np \int F^{5np-1} Q_{ij}^{kl} g_{kl} \partial_i h \partial_j h w^p \psi^2 dg \\ &\equiv T_1 + \dots + T_6. \end{aligned}$$



Let us bound  $T_i$ ,  $i = 1, \dots, 6$ . Observe that

$$|T_1| \leq 10np(5np-1)\|L\|_\infty \int F^{5np-2} |\nabla F| |\nabla h|^2 w^p \psi^2 dg.$$

Since  $|\nabla h|^2 = F$ , we deduce, using  $w^p = F^{5np}$ ,

$$\begin{aligned} |T_1| &\leq 10np(5np-1)\|L\|_\infty \int F^{5np-1} |\nabla F| w^p \psi^2 dg \\ &\leq 10np\|L\|_\infty \int |\nabla w^p| w^p \psi^2 dg. \end{aligned}$$

Thus, after a little calculation, we obtain,

$$(2.15) \quad |T_1| \leq \frac{1}{10} \int |\nabla(w^p \psi)|^2 dg + cp^2 \|L\|_\infty^2 \int w^{2p} \psi^2 dg + c \|\nabla \psi\|_\infty^2 \int_{\text{supp } \psi} w^{2p} dg.$$

Next

$$\begin{aligned} |T_2| &\leq 10np\|L\|_\infty \int F^{5np-1} |\text{Hess } h| |\nabla h| w^p \psi^2 dg \\ &\leq np \int F^{5np-1} |\text{Hess } h|^2 w^p \psi^2 dg + cnp\|L\|_\infty^2 \int F^{5np-1} |\nabla h|^2 w^p \psi^2 dg. \end{aligned}$$

Recalling again that  $|\nabla h|^2 = F$  and the definition of  $I_1$ , we deduce

$$(2.16) \quad |T_2| \leq -\frac{I_1}{10} + cnp\|L\|_\infty^2 \int w^{2p} \psi^2 dg.$$

Since  $T_3$  is similar to  $T_2$ , we also have

$$(2.17) \quad |T_3| \leq -\frac{I_1}{10} + cnp\|L\|_\infty^2 \int w^{2p} \psi^2 dg.$$

By Young's inequality

$$|T_4| \leq \frac{1}{2} \int |\nabla(w^p \psi)|^2 dg + 50n^2 p^2 \|L\|_\infty^2 \int F^{10np-2} |\nabla h|^4 \psi^2 dg.$$

Since  $F = |\nabla h|^2$  and  $w = F^{5n}$ , this shows

$$(2.18) \quad |T_4| \leq \frac{1}{2} \int |\nabla(w^p \psi)|^2 dg + cp^2 \|L\|_\infty^2 \int w^{2p} \psi^2 dg.$$

Next

$$|T_5| \leq 10np\|L\|_\infty \|\psi\|_\infty \int F^{5np-1} |\nabla h|^2 w^p \psi dg,$$

which becomes

$$(2.19) \quad |T_5| \leq 10np\|L\|_\infty \|\psi\|_\infty \int w^{2p} \psi dg.$$

Lastly

$$(2.20) \quad |T_6| \leq 10np \int F^{5np-1} |\nabla h|^2 w^p \psi^2 dg = 10np \int w^{2p} \psi^2 dg.$$

Substituting (2.15)-(2.20) into (2.14), we find that

$$(2.21) \quad |I_3| \leq \frac{|I_1|}{5} + \frac{3}{5} \int |\nabla(w^p \psi)|^2 dg + c \frac{p^2 \|L\|_\infty^2 + 1}{[(1-\sigma)r]^2} \int_{\text{supp } \psi} w^{2p} dg.$$

Here we recall that

$$I_1 = -10np \int F^{5np-1} |Hess h|^2 w^p \psi^2 dg.$$

Using the inequality

$$|Hess h|^2 \geq \frac{1}{n} (\Delta h)^2 = \frac{1}{n} |\nabla h|^4,$$

we find that

$$I_1 = \frac{I_1}{2} + \frac{I_1}{2} \leq \frac{I_1}{2} - 5p \int F^{5np-1} |\nabla h|^4 w^p \psi^2 dg$$

which induces, since  $w = F^{5n}$  and  $F = |\nabla h|^2$ , that

$$(2.22) \quad I_1 \leq \frac{I_1}{2} - 5p \int F w^{2p} \psi^2 dg.$$

Substituting (2.22), (2.21) and (2.13) into (2.12), we deduce

$$\begin{aligned} \int |\nabla(w^p \psi)|^2 dg &\leq \frac{I_1}{2} - 5p \int F w^{2p} \psi^2 dg + (\epsilon + 1) \int F w^{2p} \psi^2 dg + \epsilon^{-1} \|\nabla \psi\|_\infty^2 \int w^{2p} \psi^2 dg \\ &\quad + \frac{|I_1|}{5} + \frac{3}{5} \int |\nabla(w^p \psi)|^2 dg + c \frac{p^2 \|L\|_\infty^2 + 1}{[(1-\sigma)r]^2} \int_{supp \psi} w^{2p} dg. \end{aligned}$$

Since  $p \geq 1$ , we can take  $\epsilon = 1$  and obtain

$$\int |\nabla(w^p \psi)|^2 dg + \int (w^p \psi)^2 dg \leq c \frac{p^2 \|L\|_\infty^2 + 1}{[(1-\sigma)r]^2} \int_{supp \psi} w^{2p} dg$$

where  $c$  may have changed in value. By the Sobolev inequality in Assumption 1, this implies

$$\left( \int (w^p \psi)^{2n/(n-2)} dg \right)^{(n-2)/n} \leq c \alpha \frac{p^2 \|L\|_\infty^2 + 1}{[(1-\sigma)r]^2} \int_{supp \psi} w^{2p} dg.$$

From this, the standard Moser's iteration implies

$$\sup_{B(x, \sigma r)} w^2 \leq \frac{C(\alpha, n, \|L\|_\infty^2)}{(1-\sigma)^n r^n} \int_{B(x, r)} w^2 dg$$

for  $r, \sigma \in (0, 1]$ . Using  $w = F^{5n}$ , we arrive at

$$\sup_{B(x, \sigma r)} F \leq \left( \frac{C(\alpha, n, \|L\|_\infty^2)}{(1-\sigma)^n r^n} \int_{B(x, r)} F^{10n} dg \right)^{1/(10n)}$$

for  $r, \sigma \in (0, 1]$ . Using the volume doubling property and an algebraic trick in [LS] e.g., we deduce

$$(2.23) \quad \sup_{B(x, r/2)} F \leq \frac{C(\alpha, n, \|L\|_\infty^2)}{r^n} \int_{B(x, r)} F dg$$

for  $r, \sigma \in (0, 1]$ . Using integration by parts, it is known that

$$\int_{B(x, r)} F dg = \int_{B(x, r)} |\nabla(\ln u)|^2 dg \leq 4 \frac{|B(x, 4r)|}{r^2} \leq c r^{n-2}$$

where we have used Assumption 2. Substituting this to (2.23), we arrive at

$$\sup_{B(x,r/2)} |\nabla(\ln u)| \leq \frac{C(\alpha, n, \|L\|_\infty^2)}{r}$$

proving the proposition.  $\square$

### 3. PROOF OF THEOREMS

*Proof.* (Theorem 1.1).

For simplicity of presentation, we omit the time variable in the proof. It is also clear that we can take  $\bar{u} = 0$ .

*Step 1.*

Pick  $u \in C^\infty(\mathbf{M})$ . Since  $\Delta u = \Delta u$  and  $\bar{u} = 0$ , we have

$$u(x) = - \int_{\mathbf{M}} \Gamma_0(x, y) \Delta u(y) dg(y),$$

where  $\Gamma_0$  is the Green's function of the Laplacian on  $\mathbf{M}$ . After integration by parts, this becomes

$$u(x) = \int_{\mathbf{M}} \nabla \Gamma_0(x, y) \nabla u(y) dg(y).$$

According to Lemma 2.3, this implies

$$(3.1) \quad |u(x)| \leq C_0 \int_{\mathbf{M}} \frac{|\nabla u(y)|}{d(x, y)^{n-1}} dg(y) \equiv C_0 I_1(|\nabla u|)(x).$$

Here  $I_1$  is the Riesz potential of order 1.

We claim that there exists a constant  $C_1$ , depending only on the constant  $\kappa$  in Property A 5, such that

$$(3.2) \quad |I_1(f)(x)| \leq C_1 [M(f)(x)]^{1-(1/n)} \|f\|_1^{1/n}.$$

for all smooth function  $f$  on  $\mathbf{M}$ . Here  $M(f)$  is the Hardy-Littlewood maximal function. The proof given here is more or less the same as in the Euclidean case (p86 [Zi]), under Property A 5, i.e.  $\kappa r^n \leq |B(x, r)| \leq \kappa^{-1} r^n$ . Let  $\delta$  be a positive number, then

$$\begin{aligned} |I_1(f)(x)| &\leq \int_{B(x, \delta)} \frac{|f(y)|}{d(x, y)^{n-1}} dg + \int_{B^c(x, \delta)} \frac{|f(y)|}{d(x, y)^{n-1}} dg \\ &\leq \sum_{j=0}^{\infty} \int_{\{2^{-j-1}\delta \leq d(x, y) < 2^{-j}\delta\}} \frac{|f(y)|}{d(x, y)^{n-1}} dg + \delta^{1-n} \int_{\mathbf{M}} |f(y)| dg \\ &\leq \sum_{j=0}^{\infty} (2^{(j+1)}/\delta)^{n-1} |B(x, 2^{-j}\delta)| \frac{1}{|B(x, 2^{-j}\delta)|} \int_{B(x, 2^{-j}\delta)} |f(y)| dg + \delta^{1-n} \int_{\mathbf{M}} |f(y)| dg \\ &\leq \sum_{j=0}^{\infty} (2^{(j+1)}/\delta)^{n-1} |B(x, 2^{-j}\delta)| M(f)(x) + \delta^{1-n} \|f\|_1. \end{aligned}$$

By Property A 5,

$$|B(x, 2^{-j}\delta)| \leq \kappa^{-1} (2^{-j}\delta)^n.$$

Combining the last 2 inequalities we deduce

$$|I_1(f)(x)| \leq C \kappa^{-1} \delta M(f)(x) + \delta^{1-n} \|f\|_1$$

which implies (3.2) by taking  $\delta = [M(f)(x)/\|f\|_1]^{-1/n}$ . We remark that if  $\delta > \text{diam}(\mathbf{M})$ , then the integral  $\int_{B^c(x,\delta)} \frac{|f(y)|}{d(x,y)^{n-1}} dg$  is regarded as zero.

Since Property A 4-5 induces volume doubling property, it is well known that the maximal operator is bounded from  $L^1(\mathbf{M})$  to weak  $L^1(\mathbf{M})$ , i.e. there is a positive constant  $C_2$ , depending only on  $\kappa$  such that

$$\beta |\{x \mid M(f)(x) > \beta\}| \leq C_2 \|f\|_1,$$

for all  $\beta > 0$ . Combining this with (3.2), we obtain, for all  $\alpha > 0$ ,

$$\begin{aligned} |\{x \mid I_1(f)(x) > \alpha\}| &\leq |\{x \mid M(f)(x) > \frac{\alpha^{n/(n-1)}}{\|f\|_1^{1/(n-1)} C_1^{n/(n-1)}}\}| \\ &\leq C_2 C_1^{n/(n-1)} \|f\|_1^{1/(n-1)} \alpha^{-n/(n-1)} \|f\|_1. \end{aligned}$$

Thus

$$(3.3) \quad \alpha^{n/(n-1)} |\{x \mid I_1(f)(x) > \alpha\}| \leq C_2 C_1^{n/(n-1)} \|f\|_1^{n/(n-1)}$$

By (3.1) we have

$$|\{x \mid |u(x)| > \alpha\}| \leq |\{x \mid |I_1(\nabla u)(x)| > \alpha C_0^{-1}\}|,$$

which infers, via (3.3) with  $f = |\nabla u|$  the following statement:

if  $\bar{u} = 0$  then for all  $\alpha > 0$ , it holds

$$(3.4) \quad \alpha^{n/(n-1)} |\{x \mid |u(x)| > \alpha\}| \leq C_3 \|\nabla u\|_1^{n/(n-1)}.$$

Here  $C_3$  is a constant depending only on the controlling constants in Properties A and B.  
*Step 2.*

Now we will convert the weak type inequality (3.4) to the desired  $L^1$  Sobolev inequality, using an argument based on the idea in [FGW]. Define the sets

$$D_k = \{x \mid |u(x)| > 2^k\}, \quad k \text{ are integers.}$$

Then

$$\|u\|_p = \left( \sum_{k=-\infty}^{\infty} \int_{D_k - D_{k+1}} |u(x)|^p dg \right)^{1/p}$$

where  $p = n/(n-1)$  here and later in the proof. This shows

$$(3.5) \quad \|u\|_p \leq \left( \sum_{k=-\infty}^{\infty} 2^{(k+1)p} |D_k| \right)^{1/p} = \left( \sum_{k=-\infty}^{\infty} 2^{(k+1)p} |\{x \mid |u(x)| > 2^k\}| \right)^{1/p}.$$

Now we define

$$g_k = g_k(x) = \begin{cases} 2^{k-1}, & x \in D_k = \{x \mid |u(x)| > 2^k\}, \\ |u(x)| - 2^{k-1}, & x \in D_{k-1} - D_k = \{x \mid 2^{k-1} < |u(x)| \leq 2^k\}, \\ 0, & x \in D_{k-1}^c = \{x \mid |u(x)| \leq 2^{k-1}\}. \end{cases}$$

It is clear that  $g_k$  is a Lipschitz function such that  $0 \leq g_k \leq |u|/2$ .

Observe that

$$D_k \subset \{x \mid g_k(x) = 2^{k-1}\} \subset \{x \mid g_k(x) > 2^{k-2}\} \subset \{x \mid |g_k(x) - \bar{g}_k| > 2^{k-3}\} \cup \{x \mid \bar{g}_k > 2^{k-3}\}$$

Here  $\bar{g}_k$  is the average of  $g_k$  on  $\mathbf{M}$ . Hence

$$(3.6) \quad \begin{aligned} |D_k| &\leq |\{x \mid |g_k(x) - \bar{g}_k| > 2^{k-3}\}| + |\{x \mid \bar{g}_k > 2^{k-3}\}| \\ &\equiv T_{k1} + T_{k2}. \end{aligned}$$

Note the average of the function  $g_k - \bar{g}_k$  is 0. Thus we can apply (3.4), with  $u$  there being replaced by  $g_k - \bar{g}_k$ , to deduce

$$(3.7) \quad T_{k1} = |\{x \mid |g_k(x) - \bar{g}_k| > 2^{k-3}\}| \leq CC_3 2^{-pk} \|\nabla g_k\|_1^p.$$

To treat  $T_{k2}$ , recall that  $g_k \leq |u|/2$  which implies

$$\bar{g}_k \leq \|u\|_1 / (2|\mathbf{M}|).$$

Therefore

$$T_{k2} = |\{x \mid \bar{g}_k > 2^{k-3}\}| \leq |\{x \mid \frac{\|u\|_1}{|\mathbf{M}|} > 2^{k-2}\}|.$$

This shows that

$$(3.8) \quad T_{k2} = \begin{cases} 0, & \text{when } k > 2 + \log_2 \frac{\|u\|_1}{|\mathbf{M}|} \equiv k_0 \\ |\mathbf{M}|, & k \leq k_0. \end{cases}$$

Substituting (3.7) and (3.8) into (3.6), we deduce

$$|D_k| \leq \begin{cases} CC_3 2^{-pk} \|\nabla g_k\|_1^p, & \text{when } k > k_0 \\ CC_3 2^{-pk} \|\nabla g_k\|_1^p + |\mathbf{M}|, & k \leq k_0. \end{cases}$$

Substituting this to (3.5) and using Minkowski inequality, we obtain

$$\|u\|_p \leq C_4 \sum_{k=-\infty}^{\infty} \|\nabla g_k\|_1 + C |\mathbf{M}|^{1/p} \sum_{k=-\infty}^{[k_0]+1} 2^k.$$

Here  $[k_0]$  is the greatest integer less than or equal to  $k_0$ . Note that the supports of  $\nabla g_k$  are disjoint and  $\nabla g_k = \nabla|u|$  in the supports. Also by the definition of  $k_0$  in (3.8), we have  $2^{k_0} = 4\|u\|_1/|\mathbf{M}|$ . Hence

$$\|u\|_p \leq C_4 \|\nabla u\|_1 + C |\mathbf{M}|^{1/p} \|u\|_1 / |\mathbf{M}|,$$

which implies, since  $p = n/(n-1)$ ,

$$\|u\|_{n/(n-1)} \leq C_4 \|\nabla u\|_1 + C \frac{1}{|\mathbf{M}|^{1/n}} \|u\|_1.$$

Here  $C$  is a numerical constant. This proves Theorem 1.1.  $\square$

Next we give

*Proof of Theorem 1.2.*

We will use the method of comparison with harmonic functions, which is a widely used for proving regularity results for elliptic systems. The new input here is the gradient estimate for harmonic functions in Section 2. The proof is divided into a few steps. We again suppress the time variable and all constants depend only on the initial metric, unless stated otherwise.

*Step 1.* First we prove the following statement. *Let  $w$  be a harmonic function in the ball  $B(x_0, r)$ . There exists a constant  $C$  such that: for all  $\rho \in (0, r]$ , there holds*

$$(3.9) \quad \int_{B(x_0, \rho)} |\nabla w|^2 dg \leq C \left(\frac{\rho}{r}\right)^n \int_{B(x_0, r)} |\nabla w|^2 dg.$$

We just need to handle the case when  $\rho \leq r/2$  since the statement is trivial otherwise. Then according to (2.3)

$$\sup_{x \in B(x_0, \rho)} |\nabla w(x)|^2 \leq C_1 \frac{1}{r^n} \int_{B(x_0, r)} |\nabla w|^2 dg.$$

Therefore

$$\int_{B(x_0, \rho)} |\nabla w|^2 dg \leq |B(x_0, \rho)| C_1 \frac{1}{r^n} \int_{B(x_0, r)} |\nabla w|^2 dg,$$

which implies (3.9) by virtue of Property A 5 in Section 1.

Now we make the following claim. Let  $v$  be a smooth function and  $w$  be a harmonic function in  $B(x_0, r)$ . Then for all  $\rho \in (0, r]$ , there is a positive constant  $C$  such that

$$(3.10) \quad \int_{B(x_0, \rho)} |\nabla v|^2 dg \leq C \left(\frac{\rho}{r}\right)^n \int_{B(x_0, r)} |\nabla v|^2 dg + C \int_{B(x_0, r)} |\nabla(v - w)|^2 dg.$$

The proof is as straight forward as follows.

$$\begin{aligned} \int_{B(x_0, \rho)} |\nabla v|^2 dg &\leq 2 \int_{B(x_0, \rho)} |\nabla w|^2 dg + 2 \int_{B(x_0, \rho)} |\nabla(v - w)|^2 dg \\ &\leq C \left(\frac{\rho}{r}\right)^n \int_{B(x_0, r)} |\nabla w|^2 dg + 2 \int_{B(x_0, \rho)} |\nabla(v - w)|^2 dg, \quad \text{by (3.9),} \\ &\leq 2C \left(\frac{\rho}{r}\right)^n \int_{B(x_0, r)} |\nabla v|^2 dg + 2C \left(\frac{\rho}{r}\right)^n \int_{B(x_0, r)} |\nabla(v - w)|^2 dg + 2 \int_{B(x_0, \rho)} |\nabla(v - w)|^2 dg. \end{aligned}$$

This proves the claim

*Step 2.* Let  $u$  be the Ricci potential. In a ball  $B(x_0, r)$ , with  $r$  less than the injectivity radius, there is a local coordinate system. From the equation  $\Delta u = n - R$ , we know that the components  $u_i$  of  $\nabla u$  satisfies

$$(3.11) \quad \begin{aligned} \Delta u_i &= \frac{1}{2} R_{i\bar{j}} u_{\bar{j}} - \partial_i R, \\ R_{i\bar{j}} &= g_{i\bar{j}} - \partial_i \partial_{\bar{j}} u. \end{aligned}$$

Fixing  $i$ , let  $w$  be a harmonic function such that

$$w|_{\partial B(x_0, r)} = u_i.$$

Then the function

$$f \equiv u_i - w$$

satisfies the equation

$$\Delta f = \frac{1}{2} R_{i\bar{j}} u_{\bar{j}} - \partial_i R$$

with zero boundary value on  $\partial B(x_0, r)$ . Using  $-f$  as a test function here and using (3.11), line 2, we deduce

$$\begin{aligned}
\int_{B(x_0, r)} |\nabla f|^2 dg &= -\frac{1}{2} \int_{B(x_0, r)} R_{i\bar{j}} u_{\bar{j}} f dg + \int_{B(x_0, r)} \partial_i R f dg \\
&= -\frac{1}{2} \int_{B(x_0, r)} (g_{i\bar{j}} - \partial_i \partial_{\bar{j}} u) u_{\bar{j}} f dg + \int_{B(x_0, r)} \partial_i R f dg \\
&= -\frac{1}{4} \int_{B(x_0, r)} (\partial_i (u_{\bar{j}})^2) f dg + \int_{B(x_0, r)} \partial_i R f dg - \frac{1}{2} \int_{B(x_0, r)} g_{i\bar{j}} u_{\bar{j}} f dg \\
&= \frac{1}{4} \int_{B(x_0, r)} (u_{\bar{j}})^2 \partial_i f dg - \int_{B(x_0, r)} R \partial_i f dg - \frac{1}{2} \int_{B(x_0, r)} g_{i\bar{j}} u_{\bar{j}} f dg.
\end{aligned}$$

Here the last step is integration by parts. Since  $|f| = |u_i - w| \leq 2\|\nabla u\|_\infty$ , we obtain

$$\int_{B(x_0, r)} |\nabla f|^2 dg \leq \frac{1}{2} \int_{B(x_0, r)} |\nabla f|^2 dg + C \int_{B(x_0, r)} (\|\nabla u\|_\infty^4 + R^2 + 1) dg,$$

which shows

$$(3.12) \quad \int_{B(x_0, r)} |\nabla(u_i - w)|^2 dg \leq Cr^n,$$

where we have used the volume upper bound and Perelman's bound on the scalar curvature  $R$ . In (3.10), we take  $v = u_i$  and apply (3.12). Thus, for  $\rho \in (0, r]$ , it holds

$$\begin{aligned}
\int_{B(x_0, \rho)} |\nabla u_i|^2 dg &\leq C \left(\frac{\rho}{r}\right)^n \int_{B(x_0, r)} |\nabla u_i|^2 dg + C \int_{B(x_0, r)} |\nabla(u_i - w)|^2 dg \\
&\leq C \left(\frac{\rho}{r}\right)^n \int_{B(x_0, r)} |\nabla u_i|^2 dg + Cr^n.
\end{aligned}$$

From here, a well known iteration process (see p60 in [HL] e.g.) shows

$$\int_{B(x_0, \rho)} |\nabla u_i|^2 dg \leq C \left(\frac{\rho}{r_0}\right)^p \int_{B(x_0, r_0)} |\nabla u_i|^2 dg + C\rho^p$$

for all  $p < n$ . Here  $r_0$  is the injectivity radius at  $x_0$ . By the  $L^2$  Poincaré inequality (2.6), this implies

$$\int_{B(x_0, \rho)} |u_i - (\bar{u}_i)_{B(x_0, \rho)}|^2 dg \leq C\rho^{p+2} \frac{1}{r_0^p} \int_{B(x_0, r_0)} |\nabla u_i|^2 dg + C\rho^{p+2}$$

for all  $\rho \leq r_0/2$ . By the same argument, this inequality also holds when the center of the ball  $x_0$  is replaced by a point  $x \in B(x_0, r_0/2)$  and the radii reduced to 1/2 of the size. i.e.

$$(3.13) \quad \int_{B(x, \rho)} |u_i - (\bar{u}_i)_{B(x, \rho)}|^2 dg \leq C\rho^{p+2} \frac{1}{r_0^p} \int_{B(x, r_0/2)} |\nabla u_i|^2 dg + C\rho^{p+2}$$

for all  $\rho \leq r_0/4$ ,  $x \in B(x_0, r_0/2)$ . Multiplying the equation

$$\Delta |\nabla u|^2 = 2|Hess u|^2 + R_{i\bar{j}} u_i u_{\bar{j}} - 2\nabla R \nabla u$$

by a standard cut-off function and doing integration by parts, using the relation  $R_{i\bar{j}} = g_{i\bar{j}} - \partial_i \partial_{\bar{j}} u$  again, it is easy to see that

$$(3.14) \quad \int_{B(x, r_0)} |\nabla u_i|^2 dg \leq C r_0^{n-2}.$$

Substituting this to (3.13), we arrive at

$$\int_{B(x, \rho)} |u_i - (\bar{u}_i)_{B(x, \rho)}|^2 dg \leq C \rho^{p+2} r_0^{n-2-p} + C \rho^{p+2}$$

for all  $\rho \leq r_0/4$ ,  $x \in B(x_0, r_0/2)$ .

Now we can apply Campanato's result (see Theorem 3.1 in [HL] e.g.) to conclude that  $\nabla u \in C^\alpha(B(x_0, r_0/2))$  for all  $\alpha \in (0, 1)$ . The norm depends only on  $g(0)$ ,  $r_0$  and  $\alpha$ . We remark that this theorem was stated in the Euclidean case. However the proof applies verbatim to any metric space equipped with the volume bounds  $c r^n \leq |B(x, r)| \leq c^{-1} r^n$  for  $r \leq 1$ . □

*Remark.* Observe that (3.14) holds for all  $r_0 > 0$ . One consequence of this is the following scaling invariant a priori bound for the Ricci curvature

$$\int_{B(x, r_0)} |\text{Ric} - g|^2 dg \leq C r_0^{n-2}, \quad x \in \mathbf{M}.$$

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